

# Notes on the multiplicity of flows in the Taylor experiment

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A representative set of experimental observations is presented demonstrating a remarkably large number of distinct steady flows that all subsist on the same boundary conditions. Commentary on the significance of these and related findings re-emphasizes previous proposals about the interpretation of the Taylor experiment.

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## 1. Preliminary discussion

A somewhat radical point of view regarding the rational explanation of events observable in the famous Taylor experiment has been developed by us in a series of theoretical and experimental papers. These contributions have for the most part canvassed newly discovered properties of steady cellular flows at moderate values of the Reynolds number  $R$  (Benjamin 1976, 1978*a*, *b*, *c*; Benjamin & Mullin 1981; Mullin 1982; Mullin, Pfister & Lorenzen 1982), but some recent studies in the same spirit have dealt with unsteady flows at higher values of  $R$  (Mullin & Benjamin 1980; Benjamin 1981; Mullin *et al.* 1981). A prime contention of the previous discussions has been that although the realistic hydrodynamic problem modelling the Taylor experiment is yet unsolved in closed form, it must have a high multiplicity of isolated solutions when  $R$  lies well above the quasi-critical range wherein Taylor cells are first easily demonstrable by standard flow-visualization techniques.† Thus, as has been evinced by numerous observations on different flows that become steady asymptotically in time under the same steady boundary conditions, the system possesses various metastable states, and these are necessarily complemented by other states of steady motion that are unstable and so unobservable experimentally.

One of the topics already examined with particular care by us is the primary-flow exchange process, in which multiplicity of steady flows is plainly an essential factor. As the aspect ratio  $\Gamma$  of the fluid-filled annulus in the Taylor experiment is raised through each in a succession of critical values, a different assemblage of Taylor cells takes priority as the primary flow. This is the flow that, for a particular geometry of

† It now seems to be widely accepted that the first appearance of toroidal cells in practice is not an abrupt phenomenon, demarked precisely by a critical value of  $R$  at which a basic flow loses stability. Careful observations reveal instead an essentially continuous process: namely, as  $R$  is gradually raised through the narrow quasi-critical range, arrays of axisymmetric, counter-rotating cells spread from the ends and finally link up and become ordered prominently at the centre of a Taylor apparatus (cf. Kusnetsov *et al.* 1977; Pfister & Rehberg 1981, figure 1). But statements to the contrary still recur in the literature.

the fluid domain, is defined uniquely by the property that it is evolved by very gradual increases in  $R$  from small values. In an apparatus with fixed outer cylinder and end walls, primary flows are found to have an even number of cells in which the spiralling motion has the *normal* direction, radially inwards close to the walls in the two end cells. A stable steady flow with an even number of normally spiralling cells is not, of course, immediately cancelled as a possibility outside the range of  $\Gamma$  in which it is the primary flow. Rather, it remains realizable as a so-called *secondary mode* at sufficiently high  $R$ , but will collapse catastrophically if  $R$  is then reduced to a critical value dependent on  $\Gamma$ . The previous experiments, with a comparatively short Taylor apparatus, have also demonstrated various *anomalous modes* of stable steady flow, so called because they are always disconnected from the primary flow, remaining for all  $\Gamma$  secondary in the sense just explained. They include flows featuring an odd number of cells, which are therefore asymmetric about a central plane perpendicular to the axis of the annulus, and also flows with an even number of cells but with an abnormal direction of spiralling (see Benjamin 1978*b*; Benjamin & Mullin 1981).

Now, a reflection on the observations already reported and debated by us might be that they are peculiar to the comparatively low range of  $\Gamma$  so far covered, and perhaps also to the fastidious experimental procedures needed to reveal the complexities at issue. A simpler view of the whole topic might be taken by emphasizing those features that are observed to be at least roughly in accord with the standard theoretical model, which takes the annulus to be infinitely long, and by dismissing the inconsistencies with it as superfluous complications due to 'end effects'. Such complications should lapse into insignificance, so it is presumed according to this view,† when  $\Gamma$  is in the higher ranges common to most rehearsals of the Taylor experiment. Our own view is different! We consider that the high multiplicity and complicated parametric dependence of the complete solution set, features that have proved particularly amenable to demonstration at low  $\Gamma$ , are intrinsic to the general hydrodynamic problem when posed realistically. They become even more rather than less pertinent when phenomena observed in a long Taylor apparatus are to be construed precisely. Thus, in our view, the presence of the ends always bears crucially on what happens in a Taylor apparatus, however long. Admittedly, certain gross features of observable phenomena may be accountable more or less convincingly by appeal to the infinite theoretical model, a relevant and much explored abstraction. But a full understanding in principle of all that happens needs to accommodate the complex facts about multiplicity, the delicacy of the primary-flow process, etc. that have been tied to the realistic problem (cf. DiPrima & Swinney 1981, §6.6).

In support of the viewpoint advocated, the central purpose of these notes is to record a typical and, we intend, emphatic demonstration of high multiplicity in a moderately long Taylor apparatus. We present experimental observations on fifteen *different* stable steady flows, all of which were realized in the same fluid subject to the same geometrical and dynamical boundary conditions. Five of these flows were realizable in two different ways, as indicated by simple considerations of symmetry, and so the total number demonstrated is twenty.

According to a sure analytic argument established and variously applied by us in the previous papers, these observations imply that the time-independent hydro-

† Which view is still popular (cf. e.g. Gorman & Swinney 1982).

dynamic problem with the boundary conditions in question has at least 39 solutions. In general, if any number  $N$  of distinct steady flows are realizable, being therefore necessarily stable, the problem has at least  $2N - 1$  solutions, of which the additional  $N - 1$  represent unstable flows. Apart from the number of solutions deducible in this way, the existence of still more is indicated by theoretical arguments concerning the primary-flow selection process (see Benjamin & Mullin 1981, §2.5). It appears that each instance of the process, arising in succession as  $\Gamma$  is raised, entails the interplay of *nine* solution branches parametrized by  $R$ , which do not include any of those representing the observed flows with odd numbers of Taylor cells.

The observations to be presented exemplify a wide range of comparable ones made by us in the apparatus described below. They have been compiled systematically in order to comprise, for a representative triplet of values for the parameters  $R$ ,  $\Gamma$  and radius ratio  $\eta$ , an identification of alternatively possible steady flows that is much fuller than any previously available. We should acknowledge, however, that a variety of such direct evidence about multiplicity in the Taylor experiment is already on record. For instance, photographs of five distinct flows realized under the same conditions were included in Benjamin (1978*b*); but as  $\Gamma$  was only 3.15 in that case, exceptional dependence on end effects might be alleged. Numerical evidence of multiplicity is also available, albeit limited so far (Alziary de Roquefort & Grillaud 1978).

It is particularly relevant to recall the prior experimental results of Burkhalter & Koschmieder (1974), who investigated steady cellular flows produced subsequently to sudden starts of their fairly long Taylor apparatus. They found various distinct flows to be producible this way for given  $\eta$  and  $\Gamma$ , the selection depending on widely varied supercritical values of  $R$  (presented rather as values of Taylor number  $T$ , proportional to  $R^2$ ). In their account, however, which tends to emphasize comparisons with the idealized theory, the flows are discriminated by records of the effective wavelength attributable to the cellular arrays, rather than by a tally of cell numbers and directions of spiralling. All their findings are consistent with the general, comparatively simple interpretation now advocated, although an explanation on such lines was not explicitly recognized by Burkhalter & Koschmieder. Namely, as we see the situation, when  $R$  lies above the quasi-critical range, the real system, determined perhaps fairly crucially by  $\Gamma$ , possesses numerous but discrete metastable states of steady motion, which have  $R$ - and  $\Gamma$ -dependent domains of attraction in some function space where time-dependent motions also belong. As the outcome of the starting transients, which are likely to depend sensitively on the starting conditions and the dimensionless terminal speed  $R$  of the rotor, the system is steered into one or other domain of attraction and thereafter converges asymptotically to the respective steady flow.

In our view this situation cannot be explained tenably by the infinite theoretical model alone, and its attributes unaccountable by the abstract model are in fact pivotal to a proper explanation. We shall rephrase our view at the end, pinpointing the main reasons for it.

#### *The observations*

The fifteen steady flows demonstrated in figure 1 were produced in a Taylor apparatus with the outer cylindrical wall stationary. The fluid-filled annulus had plane end-walls, perpendicular to the axis, which were also stationary. The set of dimensionless parameters applying to all these flows is specified as follows, where  $r_1$  and  $r_2$  denote

the radii of the concentric cylindrical walls,  $l$  the length of the annulus, and  $\Omega$  the angular speed of the inner wall:

$$\text{radius ratio } \eta = r_1/r_2 = 0.600;$$

$$\text{aspect ratio } \Gamma = l/(r_2 - r_1) = 12.61;$$

$$\text{Reynolds number } R = \Omega r_1(r_2 - r_1)/\nu = 359.$$

The liquid used was an aqueous solution of glycerol with kinematic viscosity  $\nu = 3.53 \text{ mm}^2/\text{s}$  at the controlled temperature  $28.2 \text{ }^\circ\text{C}$  of the apparatus. A small amount of a pearly substance was added for visualization, and the photographs in figure 1 were taken with the flows illuminated by a beam of light collimated through a narrow slit and directed radially. The camera was placed in approximately but not exactly the same position for each photograph, so capturing slight variations in some of the incidental highlights. The fifteen different flows are all identified in the caption to the figure.

The representative value of  $R$  was chosen in the light of previous results concerning the limits of stability for normally spiralling and anomalous modes in the Taylor experiment (Mullin & Benjamin 1980).<sup>†</sup> These limits are quite high for the comparatively small value of  $\eta$  in the present and previous experiments. While being high enough to admit a satisfactorily wide variety of different steady states, this value of  $R$  is below the threshold for the onset of travelling waves in each of the cellular arrays realized.

At the chosen values of  $\eta$  and  $\Gamma$ , the primary flow comprises twelve cells. This flow could easily be produced by slowly raising  $\Omega$  from a small value to its prescribed final value. All the other, secondary modes were produced by sudden starts of the rotation in respective ranges of  $\Omega$  found by trial (see Benjamin 1978*b* for a full account of such a procedure). Except for those comprising eight, nine, seventeen and eighteen cells, these modes were obtained directly with  $\Gamma$  at its prescribed value. The other four modes furthest from the primary flow were obtained first at smaller or larger values of  $\Gamma$ , and then they were respectively stretched or compressed by very gradual adjustments of  $\Gamma$  to the required value. Long settling times were allowed to confirm that the recorded final states of motion were steady and stable.

Other than the twelve-cell primary flow, the fourteen secondary modes demonstrated in figure 1 all collapse eventually as  $R$  is reduced by gradual steps. The critical values  $R_0$  of  $R$  at which they were found to lose stability and collapse are listed in table 1. From these values and from what is already known about the dependency of  $R_0$  upon  $\Gamma$  for anomalous modes (Benjamin & Mullin 1981), it appears possible that two more stable steady flows exist for the chosen values of  $\eta$ ,  $\Gamma$  and  $R$ . These are the two flows next in order to those included in table 1, namely the eight-cell and eighteen-cell anomalous modes. Careful attempts to produce them were unsuccessful, however. Although producible with ease at values of  $\Gamma$  respectively smaller and larger than the chosen value, they did not survive the gradual adjustments whereby we tried to take them to the required value of  $\Gamma$ . The vulnerability of these modes appeared accountable to a premature onset of travelling waves, triggered by the enforced stretching or

<sup>†</sup> Note: in figure 1 of Mullin & Benjamin (1980) the values of  $R_*$  were inadvertently left based on a definition  $\Omega r_1^2/\nu$  of Reynolds number that had been used previously, rather than  $\Omega r_1(r_2 - r_1)/\nu$  as stated in the text and readopted in the present paper. The correct values of  $R_*$  are thus 0.626 times those given.

		Normally spiralling modes								
No. of cells		8	10	14	16	18				
$R_0$		233	81	75	110	197				
		Anomalous modes								
No. of cells	9	10	11	12	13	14	15	16	17	
$R_0$	270	253	244	226	190	201	205	206	212	

TABLE 1. Lower limits of stability for secondary modes at  $\eta = 0.600$ ,  $\Gamma = 12.61$

compression of the upper end-cell as  $\Gamma$  was adjusted; for in either case traces of travelling-wave behaviour was seen to precede the collapse of the mode.

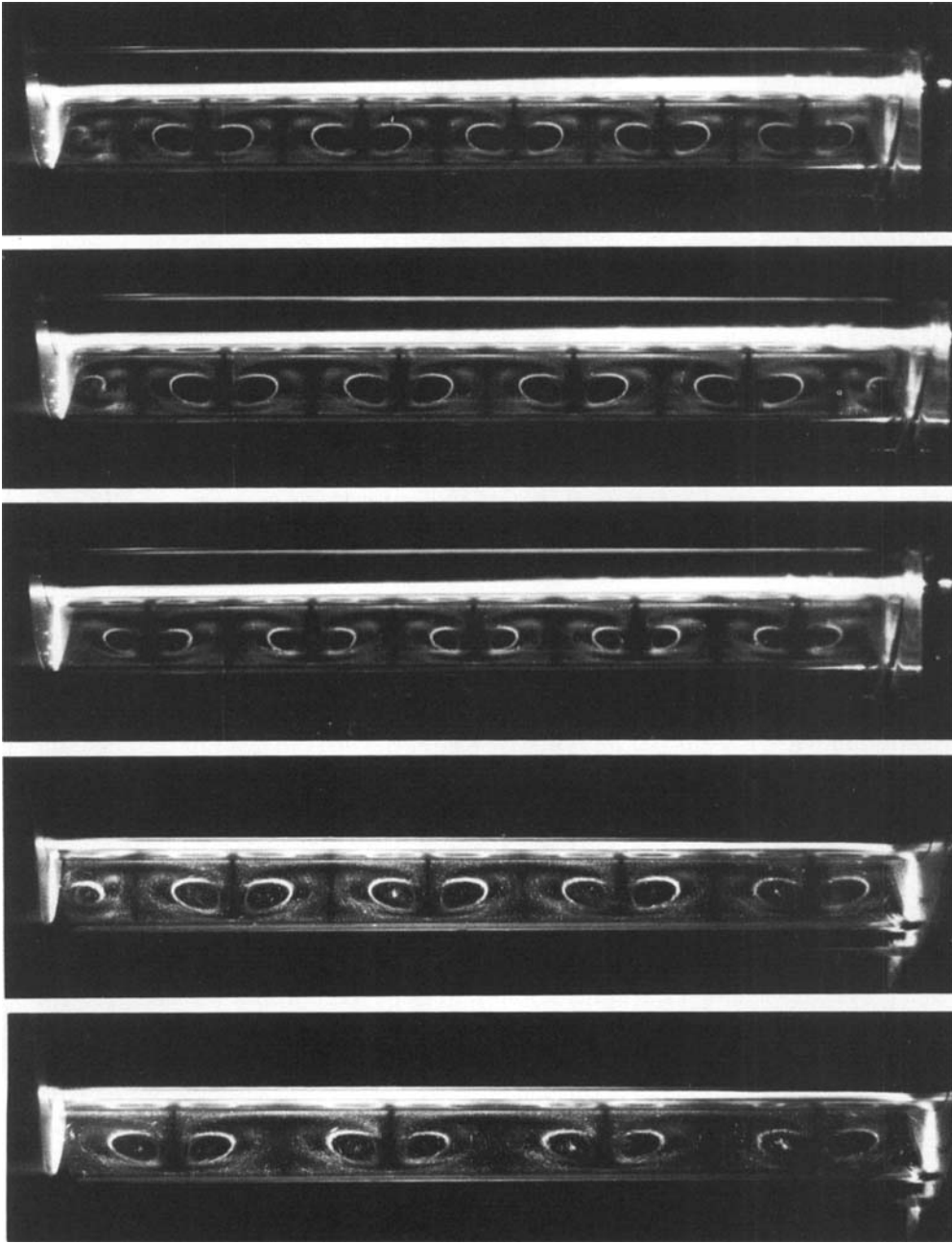
Figure 1 includes five anomalous modes with an odd number of cells (i.e. from 9 to 17), each of which flows is therefore capable of two different realizations. In one realization the abnormally spiralling end cell is at the top of the annulus, and in the other it is at the bottom, but there is complete parity between the two in the case of symmetric end-conditions (cf. Benjamin 1978*b*, p. 39). For this reason photographs of the five extra stable steady flows have not been included, but it can be asserted that the total number of stable steady flows effectively demonstrated by figure 1 is *twenty*. Thus the multiplicity of steady flows is definitely established to be at least  $2 \times 20 - 1 = 39$ , and the facts reported in the last paragraph point to the likelihood of its being at least  $2 \times 22 - 1 = 43$ .

## 2. Outline of theoretical issues

Let us now review some basic questions of interpretation that are highlighted by such evidence of high multiplicity in the Taylor experiment.

(i) *What is the relation between each of the observable flows and the periodic cellular flows described by the infinite model?* An analysis by Kogelman & DiPrima (1970), marginally refined by Nakaya (1974; see also DiPrima & Swinney 1981, pp. 154–157), should first be appreciated in connection with this question. They investigated the stability, to axisymmetric disturbances, of infinite Taylor-vortex flows at supercritical values of  $R$  (i.e.  $R > R_c(\lambda_m)$ , where  $R_c(\lambda_m) = \min R_c(\lambda)$  is the critical value for bifurcation from the axially uniform, circular Couette flow into Taylor vortices that are normalized in respect of axial translations and have the optimal wavelength  $\lambda_m$ ). On this basis they showed that, for small positive  $R - R_c(\lambda_m)$ , these periodic flows are stable if  $\lambda$  lies in an interval approximately centred on  $\lambda_m$  and whose width is approximately  $\sqrt{\frac{1}{3}}$  times the difference between the two roots of  $R = R_c(\lambda)$ . Thus, for a given  $R - R_c(\lambda_m) > 0$ , the infinite model provides a continuum of periodic solutions that may be stable, together with two adjacent continua of unstable solutions.

In certain important respects, these theoretical findings evidently conform with observations such as ours. It appears experimentally that, except near the ends in a long Taylor apparatus, cellular flows above the quasi-critical range of  $R$  closely copy those described by the infinite model, and their spacings fall within the predicted interval of stability for  $\lambda$  (= twice spacing of cells in a periodic array). This conformity is to be expected, of course, because presumably upon being disturbed the real flows



(e)

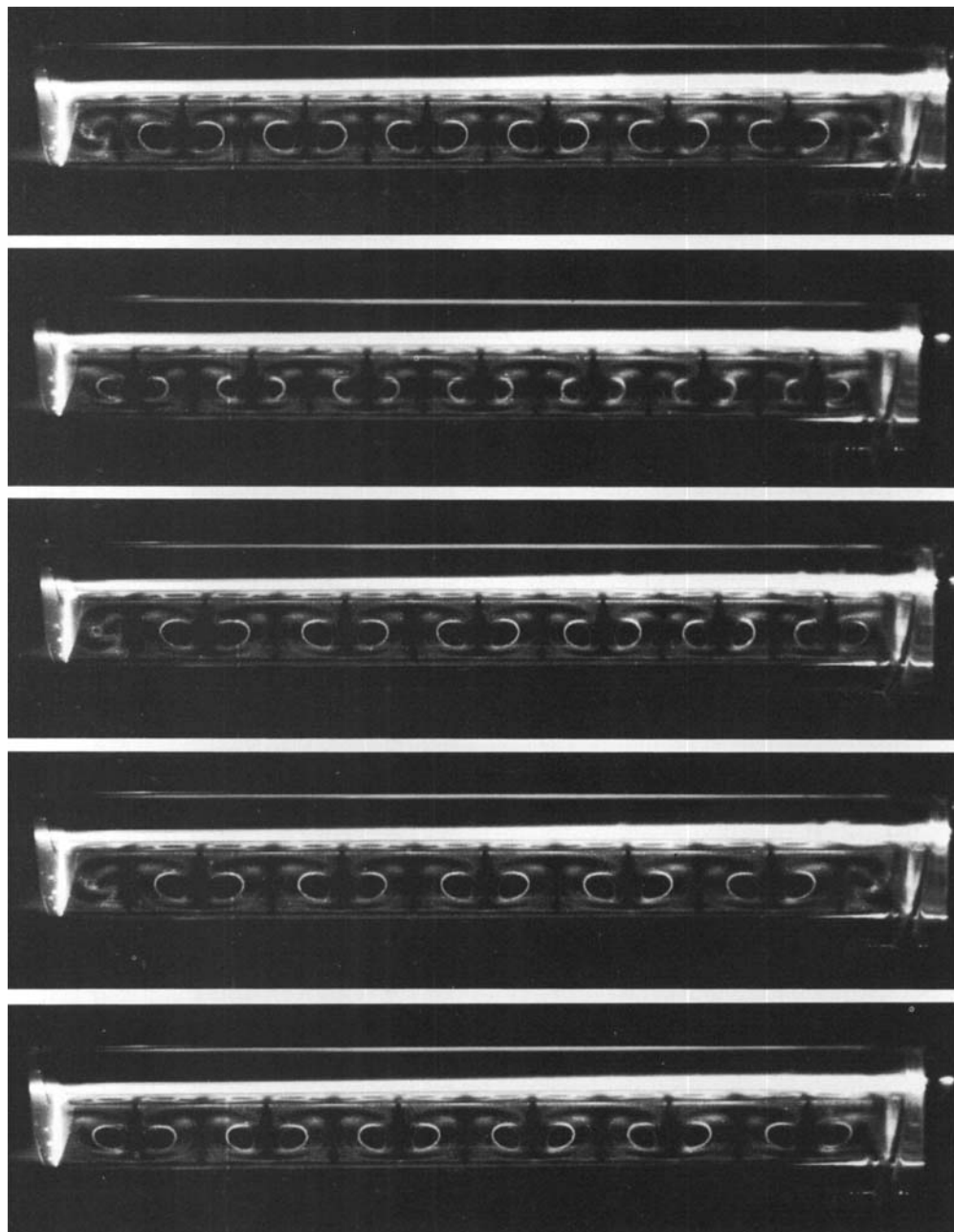
(d)

(c)

(b)

(a)

FIGURE 1(a-e). For caption see page 226.



(e)

(f)

(h)

(g)

(i)

FIGURE 1(*f-j*). For caption see page 226.

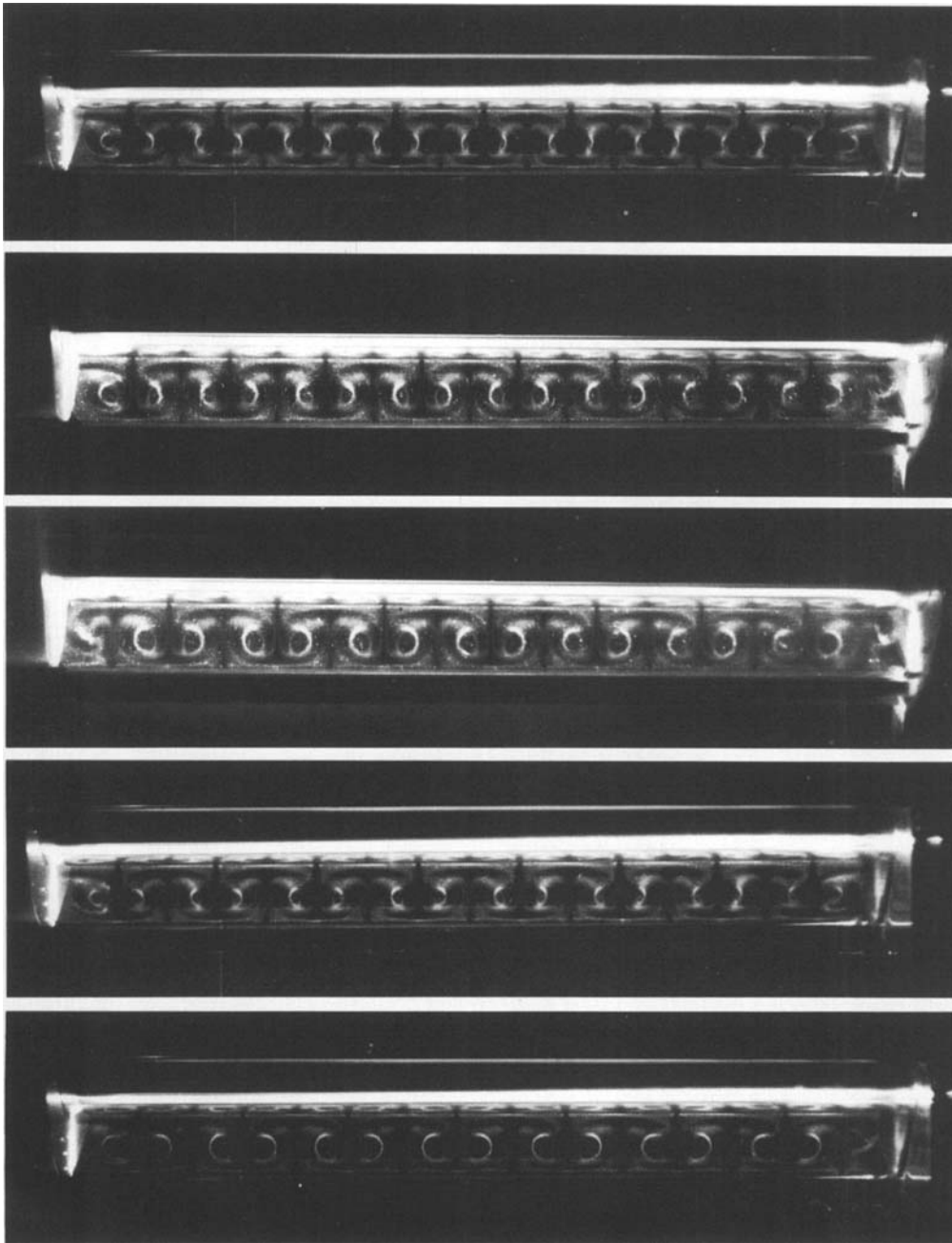


FIGURE 1. Photographs showing various cellular modes of steady flow for  $\eta = 0.600$ ,  $\Gamma = 12.61$ ,  $R = 359$ : (a) normal eight-cell; (b) nine-cell, abnormal cell at top; (c) normal ten-cell; (d) anomalous ten-cell; (e) eleven-cell, abnormal cell at top; (f) normal twelve-cell (the primary flow); (g) anomalous twelve-cell; (h) thirteen-cell, anomalous cell at top; (i) normal fourteen-cell; (j) anomalous fourteen-cell; (k) fifteen-cell, abnormal cell at bottom; (l) normal sixteen-cell; (m) anomalous sixteen-cell; (n) seventeen-cell, abnormal cell at bottom; (o) normal eighteen-cell.



will respond *locally* in nearly the same way as their periodic counterparts; thus they could not be stable if the infinite model for them were not so. Insights into this aspect are given by an approximate analysis that Stuart & DiPrima (1978) have made, considering a time-dependent envelope equation for arrays of Taylor cells at slightly supercritical  $R$ . Accordingly, it is clear that findings such as those of Kogelman & DiPrima indicate necessary rather than sufficient conditions of stability for real cellular flows. In general they cannot guarantee that such a flow, being inevitably subject at the ends to major distortions from periodicity, is realizable in a given Taylor apparatus. Reconsidering table 1, we note that the respective lower limits  $R_0$  of stability for the five secondary modes with even numbers of normally spiralling cells coincide more or less with the Kogelman–DiPrima estimate based on effective wavelength (cf. Burkhalter & Koschmieder 1974, figure 6; DiPrima & Swinney 1981, figure 6.5). But the measured values of  $R_0$  for all nine anomalous modes are too high by far to be reconcilable with the infinite model. It is impressive, for example, that whereas the fourteen-cell normal and anomalous modes are virtually identical over a central length of the arrays, the stability limits for these two flows are quite different.

As noted earlier, when  $N (> 1)$  different steady flows are known to be observable and therefore stable for given  $\eta$ ,  $\Gamma$ ,  $R$ , at least  $N - 1$  unstable flows also exist. It is tempting to suppose that these unknown flows may be related to ones in the continua of unstable periodic solutions disclosed by the Kogelman–DiPrima analysis, but this idea is yet unexplored.

(ii) *What precisely happens when a realistic model allowing for end effects is taken to the limit as  $\Gamma \rightarrow \infty$ ? Does the complete solution set, at a supercritical value of  $R$ , develop with  $\Gamma$  so that it converges in some meaningful sense to the solution set for the infinite model, recovering all its properties?* Plainly from the evidence, the multiplicity of steady flows as exemplified by the present observations will increase without bound as  $\Gamma$  is increased, and it is plausible that the profusion of discrete solutions for large but finite  $\Gamma$  (ordered, say, according to average spacing of cells) will in the limit fill out the *continuum* of alternative cellular flows provided by the infinite model. This presumption may be deceptive, however, because convergence in the considered sense will only hold in a prescribed fractional length of the annulus. For any finite  $\Gamma$  however large, major differences between the realistic and idealized models will remain at the two ends, and there is no *a priori* reason to dismiss these differences as immaterial either to the stability or to other experimentally discernible properties of the *whole* flows in question.

In fact, respective to every given finite  $\Gamma$ , a unique primary flow exists, and all the other flows possible at sufficiently high  $R$  are secondary modes having properties that are qualitatively different. As  $\Gamma$  becomes large, the difference in  $\Gamma$  spanning successive instances of the primary-flow exchange process probably tends to a constant (i.e. the wavelength  $\lambda_m$  of the optimal Taylor-vortex flow). But in principle this process is distinctive every time it operates, causing a definite switch of roles between two steady cellular modes and so entailing essential dependence on end effects.

(iii) *Can the situation for very large but finite  $\Gamma$  be given a simple interpretation in terms of perturbed bifurcation theory?* An affirmative answer has often been suggested, particularly as regards the delicate but usually continuous developments of the primary flow with increasing  $R$  whereby Taylor cells first manifest in practice. It is commonly supposed that this aspect of the Taylor experiment simplifies progressively

as  $\Gamma$  is made larger, leaving only a residual perturbation from behaviour according to the infinite model. In a rational approach to the question, however, various difficulties appear indicating that in principle more rather than less complication arises as  $\Gamma$  is greatly enlarged. On one count, observed cellular flows in the central part of a long Taylor apparatus must always be 'aware' of the ends, at least inasmuch as the ends fix the phase of the axial variations everywhere; and on another, as considered under (ii), the primary flow mutates by a complex process every time the total length, however great, is increased by about  $\lambda_m$ .

We accordingly consider it wrong (or rather right only to the extent of a rough analogy) to describe the observed situation as a simple 'softening' of the supercritical bifurcation predicted by the infinite model when the optimal wavelength is *prescribed*. In other words, end effects when  $\Gamma$  is very large are *not* rationally modelled by the canonical unfolding of a pitchfork bifurcation – that is, by the well-known algebraic characterization (cusp catastrophe) that simply and unequivocally models, for example, the perturbed Euler elastica which has just two post-buckling modes for loads slightly above the first critical value. For an extremely long Taylor apparatus, in contrast, the process determining the primary flow must discriminate among the great number of discrete modes that are close to the one selected. In principle, therefore, this process will always depend definitely and in intricate fashion on end effects, however elusive the dependence may be to demonstrate experimentally.

(iv) *How does the multiplicity of steady solutions affect observations of time-dependent phenomena at high  $R$ ?* An astounding variety of such phenomena has been recorded, notably in the extensive study by Coles (1965), and accordingly this question calls for a circumspect answer. Depending on the cellular mode established at the outset, many different routes to turbulence may be followed by gradually raising  $R$  beyond the respective limit  $R_w$  for stability of steady motion. It is noteworthy, however, that a narrower view of possible events at high  $R$  in the Taylor experiment has been taken in several more recent investigations, which have presumably concentrated on particular modes. In justification there is the fact that after a particular array of toroidal cells has been established with  $R < R_w$  and with  $\Gamma$  well inside the limits for survival of the mode, the same cellular structure may be preserved when  $R$  is raised to far above  $R_w$ , although fluctuations with progressively more complicated temporal dependence are superposed upon it. As is now well known, the typical progression consists of first singly then multiply periodic fluctuations, due to azimuthal travelling waves, and then incipient turbulence with sharp spectral peaks perhaps still prominent.

This is a standard version of events which has been reinforced in many recent commentaries. Also some progress has been made towards explaining at least the first manifestations of unsteadiness by means of the infinite model (for a recent review see DiPrima & Swinney 1981, §6.4). There are nevertheless various reasons for general caution about the interpretation of high- $R$  experiments, which considerations include the following. It is already well known that  $R_w$  is sensitive to  $\eta$  and also to  $\Gamma$ . Furthermore, in previous experiments of ours in a comparatively short apparatus,  $R_w$  was shown to be a different, highly peaked function of  $\Gamma$  for *each particular* normal or anomalous mode (Mullin & Benjamin 1980); and this behaviour seems unlikely to disappear in higher ranges of  $\Gamma$ . Coles (1965) and others including ourselves have also shown the forms of the first and subsequent wave motions to vary widely with  $\Gamma$  for a particular cellular mode, and to change for different modes. Again depending on the

particular mode, quite different turbulent regimes at high  $R$  may be observed to occur simultaneously in different parts of the same Taylor apparatus (Mullin *et al.* 1981). Mode jumping and other sudden changes induced by variations in  $R$ , often with strong hysteresis, have also been observed by Coles and others. Unquestionably, therefore, a due answer to question (iv) is that multiplicity profoundly affects the general possibilities for observations at high  $R$ , although of course exclusive selection of a cellular mode and avoidance of the more complicated possible effects may often be necessary for a tractable experimental approach.

(v) *Is profuse multiplicity liable to complicate a numerical approach to the realistic hydrodynamic problem?* Evidently it is, and therefore caution is needed, because the complex experimental facts will be duplicated by any comprehensive numerical study. Particular flows with chosen arrays of cells, satisfying realistic end-conditions, may be computed unambiguously, as exemplified in the study by Alziary de Roquefort & Grillaud (1978). But other stable and unstable solutions to the same boundary-value problem will generally lurk not far from selected ones, and it may be a difficult task to comprehend and discriminate among the complete solution set. For instance, no numerical result is yet available bearing on the primary-flow exchange process.

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